

Bethe Ansatz in Quantum Mechanics.

1. The Inverse Method of Separation of Variables ¹

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Abstract

In this paper we formulate a general method for building completely integrable quantum systems. The method is based on the use of the so-called multi-parameter spectral equations, i.e. equations with several spectral parameters. We show that any such equation, after eliminating some spectral parameters by means of the so-called inverse procedure of separation of variables can be reduced to a certain completely integrable model. Starting with exactly or quasi-exactly solvable multi-parameter spectral equations we, respectively, obtain exactly or quasi-exactly solvable integrable models.

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1 Introduction

There is a deep relationship between completely integrable multi-dimensional quantum systems and one-dimensional multi-parameter spectral equations. One way to explain this relationship is to start with completely integrable models arising in the famous r -matrix approach (see e.g. [4, 2, 3] and references therein). As it was demonstrated by Sklyanin, many of these models allow a complete separation of variables in some generalized coordinate systems which, generally, cannot be built by means of purely coordinate transformations (see e.g. [7] and references therein). The result of the separation of variables is, as a rule, a system of one-dimensional differential (or pseudo-differential) equations with several spectral parameters. One of these parameters is the energy of the initial system, while the others play the role of so-called *separation constants*. The spectral problem for the initial model reduces then to the problem of finding the admissible values for these parameters for which the separated equations have solutions of a special form. This form is dictated by the structure of the Hilbert space of the initial model and can be considered as an *ansatz* for the resulting equations. The general structure of this ansatz is closely related to the famous *Bethe Ansatz* which can be used for constructing solutions of the initial completely integrable system. It also turns out that the numerical equations determining the spectra of the multi-parameter spectral equations exactly coincide with the so-called Bethe Ansatz equations determining the spectrum of the initial model.

These facts suggest to formulate an independent approach for building completely integrable and exactly solvable quantum systems. The idea is very simple: One should start with a certain N -parameter spectral equation having an exactly constructable set of solutions and interpret N copies of it (rewritten in N different variables) as the result of separation of variables in a certain N -dimensional quantum model. Identifying one of the N spectral parameters with the energy of this N -dimensional model and considering the remaining $N - 1$ parameters as separation constants, one can easily reconstruct the form of its hamiltonian. Since the total number of spectral parameters is N , there are N different ways for identifying a spectral parameter with the energy. This leads us to N independent hamiltonians whose spectra are completely determined by the spectrum of the initial N -parameter spectral equation. It can also be shown that the resulting multi-dimensional hamiltonians commute with each other and thus form a certain N -dimensional completely integrable (and exactly solvable) system. It is worth stressing that this (inverse) method is far from being new: It appeared approximately at the same time as Sklyanin's (direct) method (see ref. [6]) and was exposed in refs. [11, 12, 13, 14] where it was called the *inverse method of separation of variables*.

With this paper we start a series of publications devoted to a detailed exposition of this inverse method of separation of variables. We intend to consider this method as a possible alternative (or rather complement) to the existing r -matrix approach. In our opinion, this method has several essential advantages in comparison with the r -matrix approach: First of all, the problem of constructing exactly solvable multi-parameter spectral equations is rather simple and leads to a very rich class of such equations. Applying to these equations the inverse method we can construct the corresponding completely integrable systems. It is easily seen that the class of systems obtained in this way is considerably richer than that obtained in the framework of the standard r -matrix approach. The second advantage lies in the fact that we can consider not only exactly but also the so-called quasi-exactly solvable multi-parameter spectral equations. These equations are distinguished by the fact that they can be solved by purely algebraic methods only for some limited parts of the spectrum but not for the whole spectrum. The completely integrable systems obtained this way also turn out to be *quasi-exactly solvable*

(about quasi-exactly solvable models and methods for their construction see e.g. the review articles [12, 8, 5, 13, 9, 1] and the book [14]). As far as we know, at present there are no methods for building such systems within the framework of the r -matrix approach. The third advantage finally is that the method for building exactly solvable multi-parameter spectral equations gives automatically the form of their solutions. This means that the problem of solving the resulting multi-dimensional completely integrable systems does not appear at all. We know these solutions from the very beginning! This is not the case for the r -matrix approach where the construction of explicit solutions for completely integrable models requires considerable efforts and is not formalised yet.

Of course, there are also some difficulties with the inverse method of separation of variables, the most obvious of which can be formulated as follows: The completely integrable models obtained by the method are formulated in those coordinate systems in which they are separable. Often it is desirable to rewrite them in a form which is typical for the r -matrix approach, i.e. as elements of the universal enveloping algebras of some Lie algebras (we think here about various spin chains and their generalizations). Unfortunately, this problem is not so easy, and, at present, it is solved only for a few cases of models.

2 Multi-parameter spectral equations

2.1 General definitions

The *multi-parameter spectral equations* play a fundamental role in our approach. We start this section giving their precise mathematical definition.

Let V_i , $i = 1, \dots, N$ be complex linear vector spaces of dimension d_i which we call *carrier spaces*. For linear operators

$$A_i \in \text{End } [V_i], \quad i = 1, \dots, N \quad (2.1)$$

and

$$B_i^j \in \text{End } [V_i], \quad j = 1, \dots, N, \quad i = 1, \dots, N \quad (2.2)$$

consider the problem of finding all possible sets of complex numbers $\lambda_1, \dots, \lambda_N$ for which the relations

$$\left[A_i - \sum_{j=1}^N B_i^j \lambda_j \right] \phi_i = 0, \quad i = 1, \dots, N \quad (2.3)$$

hold for some $V_i \ni \phi_i$, $\phi_i \neq 0$. We consider these relations as a system of simultaneous equations for the numbers $\lambda_1, \dots, \lambda_N$ and vectors ϕ_1, \dots, ϕ_N .

Definition 1.1. We call the system (2.3) a system of multi-parameter spectral equations. The numbers $\lambda_1, \dots, \lambda_N$ will be called *spectral parameters*.

It is convenient to regard the parameters $\lambda_1, \dots, \lambda_N$ as coordinates of a point in a certain affine space Λ of dimension N .

Definition 1.2. We call Λ the *spectral space* and those of its points for which the system (2.3) becomes solvable we call the *spectral points*. The set of all spectral points will be called the *spectrum* of system (2.3) and is denoted by Σ .

Obviously, the spectrum Σ can be found from the system of secular equations

$$\det_i \left[A_i - \sum_{j=1}^N B_i^j \lambda_j \right] = 0, \quad i = 1, \dots, N \quad (2.4)$$

in which \det_i denotes the determinant of operators acting in the space V_i . Since (2.4) is a system of N equations for N unknowns, the spectrum of system (2.3) is generally discrete, except for some special degenerate cases when it may be continuous or empty.

Definition 1.3. We call a system of multi-parameter spectral equations *finite - dimensional* if the corresponding carrier spaces are all finite - dimensional. Otherwise it is *infinite - dimensional*.

In the general case, the number of solutions of a finite-dimensional system of multi-parameter spectral equations is

$$N_{sol} = \prod_{i=1}^N \dim V_i \quad (2.5)$$

Indeed, if all the carrier spaces V_i are finite-dimensional, the secular equations (2.4) for the numbers $\lambda_1, \dots, \lambda_N$ become algebraic equations of orders $\dim V_i$, $i = 1, \dots, N$. The number of solutions for a system of such equations is, obviously, given by formula (2.5). If the dimensions of carrier spaces tend to infinity, the number of solutions of system (2.4) increases. In the limit $\dim V_i = \infty$ the spectrum of the multi-parameter spectral equations (2.3) becomes infinite. This leads us to the following simple result:

Proposition 1.1. Systems of finite-dimensional multi-parameter spectral equations have finite spectrum. The spectra of infinite-dimensional systems of multi-parameter spectral equations are in general infinite.

2.2 Two trivial cases

In particular cases system (2.3) reduces to well known equations of linear algebra. Consider the following two simple examples.

Example 1.1. Let $N > 1$ and $\dim V_i = 1$, $i = 1, \dots, N$. Then the operators A_i and B_i^j become numbers. The role of the vectors ϕ_i is played by non-zero numbers which can be chosen arbitrarily because of the linearity of (2.3). In this case the spectral parameters λ_i satisfy a system of inhomogeneous linear equations of the form

$$A_i - \sum_{j=1}^N B_i^j \lambda_j = 0, \quad i = 1, \dots, N. \quad (2.6)$$

If the matrix B_i^j is non-degenerate, then (2.6) has a unique solution

$$\lambda_j = \sum_{i=1}^N (B^{-1})_j^i A_i, \quad j = 1, \dots, N \quad (2.7)$$

and the spectrum of system (2.3) consists of a single spectral point.

Example 1.2. Let $N = 1$ and $\dim V_1 > 1$. We denote $V_1 = V$, $A_1 = A$, $B_1^1 = B$ and also $\phi_1 = \phi$ and $\lambda_1 = \lambda$. Then system (2.3) reduces to a single one-parameter spectral equation of the form

$$[A - B\lambda]\phi = 0, \quad V \ni \phi \neq 0. \quad (2.8)$$

If the operator B is non-degenerate, then

$$\lambda \in \text{Spec} [B^{-1}A]. \quad (2.9)$$

In this case the properties of the spectrum of system (2.3) can be analysed simply by using the standard spectral theory of linear operators in vector spaces.

The two examples demonstrate that multi-parameter spectral equations of the form (2.3) can be regarded as generalizations of both the inhomogeneous (non-spectral) and homogeneous (spectral) linear equations.

2.3 More complicated examples

It is perhaps instructive to consider two simple examples in which the features of both the inhomogeneous and homogeneous equations are simultaneously present. Let us first consider an example of finite-dimensional multi-parameter spectral equations.

Example 1.3. Let $N = 2$ and $\dim V_1 = \dim V_2 = 2$. In this case all the operators A_1 , B_1^1 , B_1^2 and A_2 , B_2^1 , B_2^2 can be represented by 2×2 matrices acting on two-dimensional vectors ϕ_1 and ϕ_2 . The two spectral parameters λ_1 and λ_2 can be found from the system of two secular equations

$$\det (A_1 - B_1^1 \lambda_1 - B_1^2 \lambda_2) = 0, \quad \det (A_2 - B_2^1 \lambda_1 - B_2^2 \lambda_2) = 0, \quad (2.10)$$

which both are algebraic equations of order two. Hence, system (2.10) has four solutions, and therefore the spectrum of the corresponding multi-parameter spectral equations consists of four spectral points. The construction of the vectors ϕ_1 and ϕ_2 associated with these spectral points can be performed in a standard way.

The next example concerns an infinite-dimensional system of multi-parameter spectral equations.

Example 1.4. Let $N = 2$. Let V_1 and V_2 be the spaces of analytic functions of a real variable x vanishing at the ends of the intervals $[a_1, b_1]$ and $[a_2, b_2]$, respectively. Then, obviously, $\dim V_1 = \dim V_2 = \infty$. Let us take

$$A_1 = A_2 = \frac{\partial^2}{\partial x^2}, \quad B_i^j = \omega_i^j(x), \quad i, j = 1, 2 \quad (2.11)$$

where $\omega_i^j(x)$ are certain analytic functions of x on the corresponding intervals. In this case, the system (2.3) takes the form

$$\begin{aligned} \left[\frac{\partial^2}{\partial x^2} - \omega_1^1(x)\lambda_1 - \omega_1^2(x)\lambda_2 \right] \phi_1(x) &= 0, & \phi_1(x) &\not\equiv 0, & \phi_1(a_1) &= \phi_1(b_1) = 0, \\ \left[\frac{\partial^2}{\partial x^2} - \omega_2^1(x)\lambda_1 - \omega_2^2(x)\lambda_2 \right] \phi_2(x) &= 0, & \phi_2(x) &\not\equiv 0, & \phi_2(a_2) &= \phi_2(b_2) = 0. \end{aligned} \quad (2.12)$$

The spectrum of this system can be found from the system of numerical equations

$$\begin{aligned}\Phi_1(\lambda_1, \lambda_2, c_1, a_1) &= 0, & \Phi_1(\lambda_1, \lambda_2, c_1, b_1) &= 0, \\ \Phi_2(\lambda_1, \lambda_2, c_2, a_2) &= 0, & \Phi_2(\lambda_1, \lambda_2, c_2, b_2) &= 0,\end{aligned}\tag{2.13}$$

in which $\Phi_1(\lambda_1, \lambda_2, c_1, x)$ and $\Phi_2(\lambda_1, \lambda_2, c_2, x)$ denote the general (normalized) solutions of equations (2.12). The system (2.13) is transcendental and generally has an infinite and discrete set of solutions. Thus the spectrum of system (2.12) is infinite and discrete.

2.4 Equivalence transformations

Let $S_i \in \text{End } [V_i]$ and $R_i \in \text{End } [V_i]$ be arbitrary invertible operators, M_i^k be the elements of an arbitrary non-degenerate $N \times N$ complex matrix and μ_i be arbitrary complex numbers. It is not difficult to see that the transformations of operators A_i and B_i^j defined as

$$\tilde{A}_i = S_i \left[A_i - \sum_{k=1}^N B_i^k \mu_k \right] R_i^{-1} \in \text{End } [V_i], \quad i = 1, \dots, N,\tag{2.14}$$

and

$$\tilde{B}_i^j = S_i \left[\sum_{k=1}^N B_i^k (M^{-1})_k^j \right] R_i^{-1} \in \text{End } [V_i], \quad j = 1, \dots, N, \quad i = 1, \dots, N,\tag{2.15}$$

and the simultaneous transformations of the spectral parameters λ_i

$$\tilde{\lambda}_j = \sum_{k=1}^N M_j^k (\lambda_k - \mu_k),\tag{2.16}$$

and vectors ϕ_i

$$\tilde{\phi}_i = R_i \phi_i\tag{2.17}$$

leave the form of equations (2.3) unchanged:

$$\left[\tilde{A}_i - \sum_{j=1}^N \tilde{B}_i^j \tilde{\lambda}_j \right] \tilde{\phi}_i = 0, \quad V_i \ni \tilde{\phi}_i \neq 0, \quad i = 1, \dots, N.\tag{2.18}$$

Definition 1.4. The transformations (2.14) – (2.17) will be called *equivalence transformations* of system (2.3). The systems (2.3) and (2.18) related by such transformations will be called *equivalent*. We shall distinguish three particular cases of such transformations:

- 1) transformations of first kind: $S_i \neq I_i$, $R_i = I_i$, $M_i^k = \delta_i^k$, $\mu_i = 0$, $i = 1, \dots, N$,
- 2) transformations of second kind: $S_i = I_i$, $R_i \neq I_i$, $M_i^k = \delta_i^k$, $\mu_i = 0$, $i = 1, \dots, N$,
- 3) transformations of third kind: $S_i = I_i$, $R_i = I_i$, $M_i^k \neq \delta_i^k$, $\mu_i \neq 0$, $i = 1, \dots, N$.

where I_i denotes the unit operator acting in the carrier space V_i .

It can be seen that the transformations of the first kind do not change the solutions of system (2.3), while the transformations of second and third kind transform the sets of spectral parameters λ_j and vectors ϕ_i , respectively. Note also that the transformations of second kind are ordinary linear transformations in the carrier spaces V_i , while the transformations of third kind are affine transformations in the spectral space Λ .

2.5 Degeneracy of the spectrum Σ

As noted above, the spectrum of multi-parameter spectral equations can be regarded as a set of points in a N -dimensional affine space Λ . The form of this set strongly depends on the concrete form of the system under consideration. In particular, it may happen that the spectrum of a certain system can be imbedded into another affine space Λ' of smaller dimension $N' < N$. In this case we shall say that this system of multi-parameter spectral equations has a *completely degenerate* spectrum. We can also imagine the situation where not the whole spectrum of a system but only some part of it can be imbedded into a space Λ' of smaller dimension. Such a spectrum will be called *partially degenerate*. Below we give a little more precise definition of degeneracy.

Definition 1.5. Let $\Sigma' \subset \Sigma$ denote some subset of the spectrum of some multi-parameter spectral equations. We call this subset *degenerate* if there exists a rectangular $K \times N$ matrix M_i^j , $i = N - K + 1, \dots, N$, $j = 1, \dots, N$ with $K < N$ linearly independent rows and a K -dimensional vector μ_i , $i = N - K + 1, \dots, N$, such that

$$\sum_{j=1}^N M_i^j (\lambda_j - \mu_i) = 0, \quad i = N - K + 1, \dots, N \quad (2.19)$$

for any $(\lambda_1, \dots, \lambda_N) \in \Sigma'$. The number K (which is equal to the rank of the matrix M_i^j) will be called the *degree* of the degeneracy. We call a degenerate subset Σ' *non-extendable* if there are no other elements of the spectrum Σ which satisfy relations (2.19) with the same M_i^j and μ_i .

It is not difficult to see that formula (2.19) describes part of an equivalence transformation of third kind described in the previous subsection (see formula (2.16)). This enables one to see that the degeneracy of a set $\Sigma' \subset \Sigma$ implies the existence of equivalence transformations of third kind which bring the values of K spectral parameters on $\Sigma' \subset \Sigma$ to zero.

Definition 1.6. We call the spectrum Σ of a system of multi-parameter spectral equations *completely degenerate* if Σ itself is degenerate and *partially degenerate* if Σ contains a finite, degenerate and non-extendable subset Σ' .

2.6 Hermitian multi-parameter spectral equations

Assume now that the carrier spaces V_i are Hilbert spaces endowed with scalar products $(\cdot, \cdot)_i$.

Definition 1.8. We call a system of multi-parameter spectral equations *hermitian* if, for all $i = 1, \dots, N$, the operators A_i and B_i^j are hermitian in the corresponding spaces V_i .

Among many nice properties of systems of hermitian multi-parameter spectral equations there are two which play an especially important role in many applications. In order to derive them assume system (2.3) to be hermitian. Then, taking the scalar product of the i th equation of this system with ϕ_i , we obtain a system of ordinary linear inhomogeneous equations

$$\bar{A}_i - \sum_{j=1}^N \bar{B}_i^j \lambda_j = 0, \quad i = 1, \dots, N \quad (2.20)$$

with $\bar{A}_i = (\phi_i, A_i \phi_i)_i$ and $\bar{B}_i^j = (\phi_i, B_i^j \phi_i)_i$. Hermiticity of the operators A_i and B_i^j implies the numbers \bar{A}_i and \bar{B}_i^j to be real. But this means that the solutions λ_j of system (2.20) also must be real. Thus we arrive at the following result:

Proposition 1.3. The spectra of the systems of hermitian multi-parameter spectral equations are real.

Let $\phi_i^{(n)}$ and $\phi_i^{(m)}$, $i = 1, \dots, N$ be two different solutions of system (2.3) with the corresponding sets of spectral parameters $\lambda_j^{(n)}$ and $\lambda_j^{(m)}$, $j = 1, \dots, N$. Then one has

$$\left[A_i - \sum_{j=1}^N B_i^j \lambda_j^{(n)} \right] \phi_i^{(n)} = 0, \quad \phi_i^{(n)} \in V_i, \quad i = 1, \dots, N \quad (2.21)$$

and

$$\left[A_i - \sum_{j=1}^N B_i^j \lambda_j^{(m)} \right] \phi_i^{(m)} = 0, \quad \phi_i^{(m)} \in V_i, \quad i = 1, \dots, N \quad (2.22)$$

Taking the scalar product of (2.21) with $\phi_i^{(m)}$ and (2.22) with $\phi_i^{(n)}$, we obtain

$$\left(\phi_i^{(m)}, A_i \phi_i^{(n)} \right)_i = \sum_{j=1}^N \left(\phi_i^{(m)}, B_i^j \phi_i^{(n)} \right)_i \lambda_j^{(n)}, \quad i = 1, \dots, N \quad (2.23)$$

and

$$\left(\phi_i^{(n)}, A_i \phi_i^{(m)} \right)_i = \sum_{j=1}^N \left(\phi_i^{(n)}, B_i^j \phi_i^{(m)} \right)_i \lambda_j^{(m)}, \quad i = 1, \dots, N. \quad (2.24)$$

Due to hermiticity of the operators A_i and B_i^j in the spaces V_i , we get by subtracting (2.23) from (2.24)

$$\sum_{j=1}^N \left(\phi_i^{(m)}, B_i^j \phi_i^{(n)} \right)_i (\lambda_j^{(m)} - \lambda_j^{(n)}) = 0, \quad i = 1, \dots, N. \quad (2.25)$$

Let us now assume that the spectral parameters are not degenerate. Then equations (2.25) can be satisfied if and only if

$$\det \left\| \left(\phi_i^{(m)}, B_i^j \phi_i^{(n)} \right)_i \right\|_{i,j=1}^N = 0. \quad (2.26)$$

Definition 1.9. We shall say that two solutions $\phi_i^{(n)}$ and $\phi_i^{(m)}$, $i = 1, \dots, N$ of system (2.3) are *orthogonal* (in the generalized sense) if they satisfy condition (2.26) which we call the *generalized orthogonality condition*.

Proposition 1.4. The solutions of a system of hermitian multi-parameter spectral equations corresponding to different spectral points are orthogonal in the generalized sense.

It is easily seen that Propositions 1.3 and 1.4 are natural generalizations of well known results in linear algebra about reality of spectra of hermitian operators in Hilbert spaces and orthogonality of eigensolutions corresponding to different eigenvalues.

3 The inverse method of separation of variables

3.1 Preliminary steps

We will show that any multi-parameter spectral equations can be reduced to a system of ordinary spectral equations with a single spectral parameter. This can be done by means of the so-called *inverse method of separation of variables* (see e.g. refs. []). To explain this consider the *composite carrier space*

$$\mathcal{V} = V_1 \otimes \dots \otimes V_N \quad (3.1)$$

and introduce the operators

$$\mathcal{A}_i = I_1 \otimes \dots \otimes A_i \otimes \dots \otimes I_N \in \text{End } [\mathcal{V}], \quad i = 1, \dots, N \quad (3.2)$$

respectively

$$\mathcal{B}_i^j = I_1 \otimes \dots \otimes B_i^j \otimes \dots \otimes I_N \in \text{End } [\mathcal{V}], \quad j = 1, \dots, N, \quad i = 1, \dots, N \quad (3.3)$$

acting in \mathcal{V} . Here as before $I_i \in \text{End } [V_i]$ denotes the unit operator in V_i . Taking

$$\varphi = \phi_1 \otimes \dots \otimes \phi_N \in \mathcal{V}, \quad (3.4)$$

we can rewrite system (2.3) in the form

$$\left[\mathcal{A}_i - \sum_{j=1}^N \mathcal{B}_i^j \lambda_j \right] \varphi = 0, \quad \mathcal{V} \ni \varphi \neq 0, \quad i = 1, \dots, N. \quad (3.5)$$

This means that any solution of the initial system (2.3) generates a solution of system (3.5).

Now consider the $N \times N$ matrix \mathcal{B}_i^j in equation (3.5). From its definition (3.3) it follows that the entries of this matrix belonging to different rows labeled by the index i commute with each other. Hence, the matrix determinant of this operator-valued matrix

$$\rho = \sum_{j_1, \dots, j_N=1}^N \varepsilon_{j_1, \dots, j_N} \mathcal{B}_1^{j_1} \cdot \dots \cdot \mathcal{B}_N^{j_N} \quad (3.6)$$

is well defined and does not depend on the order of the factors in formula (3.6). For this determinant, which defines obviously an operator, one can write

$$\delta_k^j \rho = \sum_{i=1}^N \Delta \mathcal{B}_k^i \mathcal{B}_i^j \quad (3.7)$$

with

$$\Delta \mathcal{B}_j^i = \sum_{j_1, \dots, j_{i-1}, j_{i+1}, \dots, j_N=1}^N \varepsilon_{j_1, \dots, j_{i-1}, j, j_{i+1}, \dots, j_N} \mathcal{B}_1^{j_1} \cdot \dots \cdot \mathcal{B}_{i-1}^{j_{i-1}} \mathcal{B}_{i+1}^{j_{i+1}} \cdot \dots \cdot \mathcal{B}_N^{j_N} \quad (3.8)$$

the cofactors of the elements \mathcal{B}_i^j . Acting by this operator matrix (3.8) on both sides of (3.5), and introducing the notations

$$\mathcal{C}_j = \sum_{i=1}^N \Delta \mathcal{B}_j^i \mathcal{A}_i, \quad i = 1, \dots, N \quad (3.9)$$

we obtain

$$\mathcal{C}_j \varphi = \lambda_j \rho \varphi, \quad \varphi \in \mathcal{V}, \quad j = 1, \dots, N \quad (3.10)$$

Hence, we arrive at the following result.

Proposition 2.1. Any solution of the system of multi-parameter spectral equations (2.3) generates a certain solution of the system of one-parameter spectral equations (3.10).

It is not difficult to see that the transition from (2.3) to (3.10) essentially solves the inverse problem of separation of variables in a very abstract formulation. The system of “one-dimensional” N -parameter spectral equations (2.3) is interpreted as a system of equations appearing after the separation of “variables” in a certain “ N -dimensional” one-parameter spectral equation. One of the N spectral parameters entering system (2.3), say parameter λ_j , is identified with a single spectral parameter of this equation. The $N - 1$ remaining spectral parameters λ_k , $k \neq j$ take the role of separation constants. The reconstruction of the form of the separable equation is performed by eliminating the separation constants from system (2.3). Since the index j may take N different values from 1 to N , we, in fact, can associate with system (2.3) N different separable equations which are listed in formula (3.10).

Definition 2.1. The equations (3.10) will be called the *universal separable equations*. The operator ρ in this equation will be called the *weight operator*.

The universality of equations (3.10) is reflected by the fact that they can be derived for *arbitrary* systems of multi-parameter spectral equations (2.3) irrespective of the concrete form and properties of the operators A_i and B_i^j forming this system.

3.2 Transformation properties of the universal separable equations

In this subsection we derive the form of the universal separable equations (3.10) constructed from the transformed equations (2.18). Introducing the operators

$$\mathcal{S} = S_1 \otimes \dots \otimes S_N \in \text{End} [\mathcal{V}], \quad \mathcal{R} = R_1 \otimes \dots \otimes R_N \in \text{End} [\mathcal{V}], \quad (3.11)$$

and using formulas (3.6) – (3.9), it is not difficult to check that the transformed version of system (3.10) reads

$$\tilde{\mathcal{C}}_j \tilde{\varphi} = \tilde{\lambda}_j \tilde{\rho} \tilde{\varphi}, \quad \tilde{\varphi} \in \mathcal{V}, \quad j = 1, \dots, N \quad (3.12)$$

where

$$\tilde{\mathcal{C}}_j = \mathcal{S} \left[\sum_{i,k=1}^N M_j^k \left(\Delta \mathcal{B}_k^i A_i - \mu_j I \right) \right] \mathcal{R}^{-1}, \quad (3.13)$$

$$\tilde{\rho} = \mathcal{S} \rho \mathcal{R}^{-1}, \quad (3.14)$$

and

$$\tilde{\lambda}_j = \sum_{k=1}^N M_j^k (\lambda_k - \mu_k), \quad (3.15)$$

respectively

$$\tilde{\varphi} = \mathcal{R} \varphi. \quad (3.16)$$

Here we denoted by I the unit operator in the composite carrier space \mathcal{V} .

3.3 Transition to completely integrable models

Up to now we did not assume any special properties for the operators A_i and B_i^j in (2.1) and (2.2). In this section we show, that there exists a wide class of these operators for which the corresponding universal separable equations (3.10) can be interpreted as spectral equations for a certain completely integrable quantum system. We start with the following definition:

Definition 2.2. We call the system of multi - parameter spectral equations non - degenerate if its weight operator ρ defined by formula (3.6) is non-singular, i.e. if $\det \rho \neq 0$.

Invertibility of the weight operator implies the existence of operators

$$\mathcal{D}_i = \rho^{-1} \mathcal{C}_i \in \text{End } [\mathcal{V}], \quad i = 1, \dots, r \quad (3.17)$$

whose eigenvalues in the space \mathcal{V} coincide with the values of the parameters λ_i :

$$\mathcal{D}_j \varphi = \lambda_j \varphi, \quad \varphi \in \mathcal{V}, \quad j = 1, \dots, N \quad (3.18)$$

From (3.18) it immediately follows that

$$[\mathcal{D}_j, \mathcal{D}_k] \varphi = 0, \quad \varphi \in \mathcal{V}, \quad j, k = 1, \dots, N \quad (3.19)$$

for any φ satisfying equations (3.18). This means that the operators \mathcal{D}_j commute with each other in the weak sense, i.e. on the states φ built from the solutions of the initial system (2.3). Commutativity in the strong sense does not follow generally from (3.19) if it is not *a priori* known that the set of these states is complete in \mathcal{V} .

Fortunately, there exists a special case in which strong commutativity of operators \mathcal{D}_j can be guaranteed without any knowledge of the completeness of φ 's. This case happens when the operators B_i^j , $j = 1, \dots, N$ commute with each other for fixed $i = 1, \dots, N$:

$$[B_i^j, B_i^k] = 0, \quad i, j, k = 1, \dots, N. \quad (3.20)$$

Definition 2.3. We call a system of multi-parameter spectral equations *commutative* if the operators B_i^j in (2.2) satisfy relations (3.20).

In order to demonstrate that commutativity of system (2.3) implies commutativity of the operators \mathcal{D}_i in (3.17), let us first note that if the operator ρ is invertible, the operators \mathcal{D}_j can be defined as solutions of the system of linear operator equations

$$\mathcal{A}_i = \sum_{j=1}^N \mathcal{B}_i^j \mathcal{D}_j, \quad i = 1, \dots, N. \quad (3.21)$$

From definitions (3.2) and (3.3) of the operators \mathcal{A}_i and \mathcal{B}_i^j and formula (3.20) it follows that

$$[\mathcal{A}_i, \mathcal{A}_k] = 0, \quad \text{for any } i, k, \quad (3.22)$$

$$[\mathcal{B}_i^j, \mathcal{B}_k^l] = 0, \quad \text{for all } i, k, j, l \quad (3.23)$$

$$[\mathcal{A}_i, \mathcal{B}_k^j] = 0, \quad \text{for any } i \neq k \text{ and any } j. \quad (3.24)$$

Taking into account relation (3.21), we get the following chain of equalities:

$$\begin{aligned}
\sum_{n,m=1}^N \mathcal{B}_i^n \mathcal{B}_i^m [\mathcal{D}_n, \mathcal{D}_m] &= \sum_{n,m=1}^N \mathcal{B}_i^n [\mathcal{D}_n, \mathcal{B}_i^m] \mathcal{D}_m + \sum_{n,m=1}^N \mathcal{B}_i^m [\mathcal{B}_i^n, \mathcal{D}_m] \mathcal{D}_n + \\
+ \sum_{n,m=1}^N [\mathcal{B}_i^n, \mathcal{B}_i^m] \mathcal{D}_n \mathcal{D}_m + \sum_{n,m=1}^N \mathcal{B}_i^n \mathcal{B}_i^m [\mathcal{D}_n, \mathcal{D}_m] &= \sum_{n,m=1}^N [\mathcal{B}_i^n \mathcal{D}_n, \mathcal{B}_i^m \mathcal{D}_m] = \\
&= [\mathcal{A}_i, \mathcal{A}_i] = 0.
\end{aligned} \tag{3.25}$$

Here we used the fact that the last two terms in the first line of (3.25) cancel and the first term in the second line vanishes because of the commutation relation (3.23). Now, using relations (3.22) and (3.24), we can consider an analogous chain for $i \neq k$:

$$\begin{aligned}
\sum_{n,m=1}^N \mathcal{B}_i^n \mathcal{B}_k^m [\mathcal{D}_n, \mathcal{D}_m] &= \sum_{n,m=1}^N \mathcal{B}_i^n \mathcal{B}_k^m [\mathcal{D}_n, \mathcal{D}_m] + \sum_{m=1}^N [\mathcal{B}_i^0, \mathcal{B}_k^m] \mathcal{D}_m = \\
= \sum_{n,m=1}^N \mathcal{B}_i^n \mathcal{B}_k^m [\mathcal{D}_n, \mathcal{D}_m] + \sum_{n,m=1}^N [\mathcal{B}_i^n \mathcal{D}_n, \mathcal{B}_k^m] \mathcal{D}_m &= \sum_{n,m=1}^N \mathcal{B}_i^n \mathcal{B}_k^m [\mathcal{D}_n, \mathcal{D}_m] + \\
+ \sum_{n,m=1}^N \mathcal{B}_i^n [\mathcal{D}_n, \mathcal{B}_k^m] \mathcal{D}_m &= \sum_{n,m=1}^N \mathcal{B}_i^n [\mathcal{D}_n, \mathcal{B}_k^m \mathcal{D}_m] = \sum_{n=1}^N \mathcal{B}_i^n [\mathcal{D}_n, \mathcal{A}_k] + \\
+ \sum_{n=1}^N [\mathcal{B}_i^n, \mathcal{B}_k^0] \mathcal{D}_n &= \sum_{n=1}^N [\mathcal{B}_i^n \mathcal{D}_n, \mathcal{A}_k] = [\mathcal{A}_i, \mathcal{A}_k] = 0
\end{aligned} \tag{3.26}$$

Combining (3.25) and (3.26), we obtain finally

$$\sum_{n,m=1}^N \mathcal{B}_i^n \mathcal{B}_k^m [\mathcal{D}_n, \mathcal{D}_m] = 0, \quad \text{for any } i \text{ and } k. \tag{3.27}$$

Now note that the invertibility of the operator ρ implies invertibility of the matrix operator (3.3). Using this fact, we obtain the relations

$$[\mathcal{D}_n, \mathcal{D}_m] = 0, \quad \text{for any } n \text{ and } m, \tag{3.28}$$

i.e. commutativity of the operators \mathcal{D}_i in the strong sense. Thus, we proved the following theorem:

Theorem 2.2. If the system of multi-parameter spectral equations (2.3) is non-degenerate and commutative then the operators \mathcal{D}_i , $i = 1, \dots, N$ defined by formulas (3.17) form a commutative family and can be considered as integrals of motion of a certain completely integrable quantum system. These operators have a common set of eigenfunctions φ constructed from the solutions of system (2.3) and their eigenvalues coincide with the values of the spectral parameters λ_i , $i = 1, \dots, N$ of equation (2.3).

3.4 Transformation properties of completely integrable equations

It is quite obvious that equivalence transformations do not affect the invertibility properties of the operator ρ , as follows from formula (3.14). However, these transformations may change the

commutativity properties of the operators B_i^j . In other words, the commutativity of a certain system of multi-parameter spectral equations does not automatically imply the commutativity of its transformed version. However, it turns out that the transformed version of equations (3.18) describes again a certain completely integrable system. Indeed, applying to (3.18) some equivalence transformation, we obtain

$$\tilde{\mathcal{D}}_j \tilde{\varphi} = \tilde{\lambda}_j \tilde{\varphi}, \quad \tilde{\varphi} \in \mathcal{V}, \quad j = 1, \dots, N \quad (3.29)$$

where the transformed operators $\tilde{\mathcal{D}}_j$, using formulas (3.17), (3.13) and (3.14), read as

$$\tilde{\mathcal{D}}_j = \mathcal{R} \left[\sum_{k=1}^N M_j^k (\mathcal{D}_k - \mu_k I) \right] \mathcal{R}^{-1} \quad (3.30)$$

But from (3.28) and (3.30) it immediately follows that for any n and m

$$[\tilde{\mathcal{D}}_n, \tilde{\mathcal{D}}_m] = 0, \quad (3.31)$$

i.e. the transformed model is again integrable. This leads us to an important conclusion which we formulate in the form of the following theorem:

Theorem 2.3. Let a system of multi-parameter spectral equations be non-degenerate and equivalent to a certain commutative system. Then the operators \mathcal{D}_i , $i = 1, \dots, N$ defined by formulas (3.17) form a commutative family and can be considered as integrals of motion of a certain completely integrable quantum system.

3.5 Hermitian completely integrable systems

Let us now consider a system of hermitian multi-parameter spectral equations. Remember that hermiticity means that the spaces V_i are endowed with scalar products $(\cdot, \cdot)_i$ with respect to which the operators A_i and B_i^j are hermitian in V_i . Then formulas (3.1) allow one to introduce in a natural way a scalar product $\langle \cdot, \cdot \rangle$ in the space \mathcal{V} . The operators \mathcal{B}_i^j will obviously be hermitian in \mathcal{V} . Using the fact that the operator \mathcal{B}_i commutes with all the operators \mathcal{B}_k^j with $k \neq i$ and observing the absence of \mathcal{B}_i^n in expression (3.8), we can conclude

$$[\Delta \mathcal{B}_j^i, \mathcal{B}_i] = 0, \quad \text{for any } j \text{ and } i. \quad (3.32)$$

Note also that, due to the commutativity of the operators \mathcal{B}_i^j and their hermiticity in V_i , the operators (3.6) and (3.8) are hermitian in \mathcal{V} :

$$(\det \mathcal{B})^+ = \det \mathcal{B}, \quad (\bar{\mathcal{B}}_j^i)^+ = \bar{\mathcal{B}}_j^i. \quad (3.33)$$

From (3.32) and (3.33) it follows that the operators \mathcal{C}_j and ρ are hermitian in the space \mathcal{V} .

Definition 2.4. We call a system of multi-parameter spectral equations *positive definite* if it has a positive definite weight operators.

It is obvious that positive definiteness of a hermitian operator ρ implies its invertibility and also the existence of the square root $\rho^{1/2}$ which can also be considered an hermitian operator. Let ρ be a positive definite hermitian operator. Then, after introducing new vectors Φ as

$$\Phi = \rho^{1/2} \varphi, \quad (3.34)$$

and new operators $\hat{\mathcal{D}}_j$ as

$$\hat{\mathcal{D}}_j = \rho^{-1/2} \mathcal{C}_j \rho^{-1/2}, \quad (3.35)$$

equation (3.10) takes the form

$$\hat{\mathcal{D}}_j \Phi = \lambda_j \Phi, \quad \Phi \in \mathcal{V}, \quad j = 1, \dots, N \quad (3.36)$$

The operators $\hat{\mathcal{D}}_j$ are obviously hermitian in the space \mathcal{V} and their eigenvalues in \mathcal{V} coincide with the admissible values of the spectral parameters λ_j . It is not difficult to see that the operators (3.35) are related to the operators introduced in (3.17) by the formula

$$\hat{\mathcal{D}}_j = \rho^{1/2} \mathcal{D}_j \rho^{-1/2}. \quad (3.37)$$

From this formula it follows that commutativity of the operators \mathcal{D}_j implies commutativity of the operators $\hat{\mathcal{D}}$. But from previous results we know that commutativity of the operators \mathcal{D} is implied by commutativity of the initial system of multi-parameter spectral equations. Collecting all these facts and recalling the definition of exact and quasi-exact solvability we arrive at the following important theorem:

Theorem 2.4. Let a system of multi-parameter spectral equations be hermitian, positive definite and commutative. Then the operators $\hat{\mathcal{D}}_j$, $j = 1, \dots, N$ are hermitian, form a commutative family, $[\hat{\mathcal{D}}_j, \hat{\mathcal{D}}_k] = 0$, and thus, can be considered as “hamiltonians” of a certain completely integrable quantum system. If, in addition, the system of multi-parameter spectral equations is exactly or quasi-exactly solvable, then the resulting completely integrable system also will be exactly or quasi-exactly solvable.

This theorem will play a central role in our further considerations. Before completing this section, it is important to discuss the case when the initial multi-parameter spectral equation becomes degenerate.

3.6 The degenerate case

Assume that the operator ρ is non-negative definite and hence singular in \mathcal{V} . In this case, for any $j = 1, \dots, N$ there exists an orthogonal decomposition of the space \mathcal{V} of the form

$$\mathcal{V} = \mathcal{V}^1 \oplus \mathcal{V}_j^2 \oplus \mathcal{V}_j^3 \quad (3.38)$$

in which the operators $\hat{\mathcal{C}}_j$ and ρ take the following block form

$$\mathcal{C}_j = \begin{pmatrix} \mathcal{C}_j^{11} & \mathcal{C}_j^{12} & \mathcal{C}_j^{13} \\ \mathcal{C}_j^{21} & \mathcal{C}_j^{22} & 0 \\ \mathcal{C}_j^{31} & 0 & 0 \end{pmatrix}, \quad \rho = \begin{pmatrix} r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.39)$$

Here $r \in \text{End}[\mathcal{V}^1]$ is a positive definite hermitian operator, $\mathcal{C}_j^{11} \in \text{End}[\mathcal{V}^1]$ and $\mathcal{C}_j^{22} \in \text{End}[\mathcal{V}_j^2]$ are hermitian operators, and $\mathcal{C}_j^{21} \in \text{Hom}[\mathcal{V}_j^2, \mathcal{V}^1]$ and $\mathcal{C}_j^{31} \in \text{Hom}[\mathcal{V}_j^3, \mathcal{V}^1]$ are hermitian conjugates of the operators $\mathcal{C}_j^{12} \in \text{Hom}[\mathcal{V}^1, \mathcal{V}_j^2]$ and $\mathcal{C}_j^{13} \in \text{Hom}[\mathcal{V}^1, \mathcal{V}_j^3]$. The decomposition (3.38) can always be chosen in such a way that all the operators \mathcal{C}_j^{22} are invertible. Writing

$$\varphi = \begin{pmatrix} \varphi^1 \\ \varphi_j^2 \\ \varphi_j^3 \end{pmatrix} \quad (3.40)$$

with $\varphi^1 \in \mathcal{V}^1$, $\varphi_j^2 \in \mathcal{V}_j^2$ and $\varphi_j^3 \in \mathcal{V}_j^3$, we can rewrite equations (3.10) in the form

$$\left. \begin{aligned} \mathcal{C}_j^{11}\varphi^1 + \mathcal{C}_j^{12}\varphi_j^2 + \mathcal{C}_j^{13}\varphi_j^3 &= \lambda_j r \varphi^1 \\ \mathcal{C}_j^{21}\varphi^1 + \mathcal{C}_j^{22}\varphi_j^2 &= 0 \\ \mathcal{C}_j^{31}\varphi^1 &= 0 \end{aligned} \right\}, \quad j = 1, \dots, N. \quad (3.41)$$

From the last equation in (3.41) it follows that

$$\varphi^1 = P\varphi' \quad (3.42)$$

where P is a certain operator such that

$$\mathcal{C}_j^{31}P = 0, \quad P^+\mathcal{C}_j^{13} = 0, \quad j = 1, \dots, N \quad (3.43)$$

and $\varphi' \in \mathcal{V}'$ is a vector belonging to a certain subspace $\mathcal{V}' \subset \mathcal{V}^1$. Substituting (3.42) into the second equation in (3.41) and using (3.43) we obtain

$$\varphi_j^2 = -(\mathcal{C}_j^{22})^{-1}\mathcal{C}_j^{21}P\varphi' \quad (3.44)$$

Substituting (3.42) and (3.44) into the first equation in (3.41) and acting on the resulting relation by the operator P^+ we obtain the following set of one-parameter spectral equations

$$\mathcal{C}'_j\varphi' = \lambda_j\rho'\varphi', \quad \varphi' \in \mathcal{V}' \quad (3.45)$$

in which

$$\mathcal{C}'_j = P^+ [\mathcal{C}_j^{11} - \mathcal{C}_j^{12}(\mathcal{C}_j^{22})^{-1}\mathcal{C}_j^{21}] P \quad (3.46)$$

are hermitian operators in \mathcal{V}' and

$$\rho' = P^+RP \quad (3.47)$$

is hermitian and positive definite in \mathcal{V}' . Noting that $(\rho')^{1/2}$ is an invertible hermitian operator and introducing the vectors,

$$\Phi' = (\rho')^{1/2}\varphi', \quad (3.48)$$

and the operators

$$\hat{\mathcal{D}}'_j = (\rho')^{-1/2}\mathcal{C}'_j(\rho')^{-1/2}, \quad (3.49)$$

we can reduce equations (3.45) to the form

$$\hat{\mathcal{D}}'_j\Phi' = \lambda_j\Phi', \quad \Phi' \in \mathcal{V}', \quad j = 1, \dots, N \quad (3.50)$$

The operators $\hat{\mathcal{D}}'_j$ are obviously hermitian in the space \mathcal{V}' and their eigenvalues in \mathcal{V}' coincide with the admissible values of the spectral parameters λ_j . As before, if the set of solutions Φ' is complete in \mathcal{V}' , the operators $\hat{\mathcal{D}}'_j$, $j = 1, \dots, N$ form a commutative family, $[\hat{\mathcal{D}}'_j, \hat{\mathcal{D}}'_k] = 0$, and thus, can be considered as “hamiltonians” of a certain completely integrable quantum system. Unfortunately, at present we do not have more general criteria for commutativity of the operators \mathcal{D}'_j in the degenerate case.

4 Conclusion. Exact and quasi-exact solvability

Let us now discuss some solvability properties of systems of multi-parameter spectral equations. First note that if all the carrier spaces V_i are finite-dimensional, then the solution of the system of multi-parameter spectral equations is a purely algebraic problem (see e.g. examples 1.1 and 1.3). In the case of infinite-dimensional carrier spaces an algebraic solution of the system of multi-parameter spectral equations is, as a rule, impossible (see e.g. example 1.4). However, the presence of some high symmetry in the system may turn it algebraically solvable, despite the infinite-dimensionality of the carrier spaces.

Definition 4.1. We call an infinite-dimensional system of multi-parameter spectral equations *exactly solvable* if all its solutions can be constructed algebraically. We call an infinite-dimensional system of multi-parameter spectral equations *quasi-exactly solvable* if only a certain finite part of its solutions allows an algebraic construction.

Consider an infinite-dimensional system of N -parameter spectral equations (2.3) which is known to be exactly solvable. Assume that the spectrum of this system is partially degenerate and thus, by definition, contains at least one finite, degenerate and non-extendable subset Σ' . Denoting its degree of degeneracy by K , one can write

$$\left[A_i - \sum_{j=1}^N B_i^j \lambda_j \right] \phi_i = 0, \quad V_i \ni \phi_i \neq 0, \quad (\lambda_1, \dots, \lambda_N) \in \Sigma', \quad i = 1, \dots, N. \quad (4.1)$$

According to the results of the previous subsection, the degeneracy allows to apply to relations (4.1) an equivalence transformation, which turns them into the form

$$\left[\tilde{A}_i - \sum_{j=1}^{N'} \tilde{B}_i^j \tilde{\lambda}_j \right] \tilde{\phi}_i = 0, \quad V_i \ni \tilde{\phi}_i \neq 0, \quad (\lambda_1, \dots, \lambda_{N'}) \in \tilde{\Sigma}', \quad i = 1, \dots, N', \quad (4.2)$$

Here we used $\tilde{\lambda}_j = 0$ for $j = N - K + 1, \dots, N$, suppressed the last K equations for $i = N - K + 1, \dots, N$ and introduced the notation $N' = N - K$. We also denoted by $\tilde{\Sigma}'$ the set of the transformed spectral points with the zero components suppressed. It is easily seen that system (4.2) is nothing else but the system of N' -parameter spectral equations whose spectrum contains by construction a certain finite exactly solvable part $\tilde{\Sigma}'$. Since the transformed system (4.2) is infinite-dimensional, its total spectrum $\tilde{\Sigma}$ is infinite. This means that the N' -parameter spectral equation obtained this way is quasi-exactly solvable. Thus we arrive at the following theorem:

Theorem 4.2. Any infinite-dimensional and exactly solvable system of N -parameter spectral equations with partially degenerate spectrum can be reduced to a certain quasi-exactly solvable N' -parameter spectral equation. Here $N' = N - K$ where K denotes the degree of the degeneracy of the spectrum of the initial equations.

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